Anti-deSitter gravitational collapse

V. Husain[b], G. Kunstatter[♯], B. Preston[♯] and M. Birukou[♯]

[b]Dept. of Mathematics and Statistics
University of New Brunswick,
Fredericton, N.B. Canada E3B 1S5.
[♯]Dept. of Physics and Winnipeg Institute of Theoretical Physics
University of Winnipeg, Winnipeg,
Manitoba Canada R3B 2E9.
(Dated: February 7, 2008)

We describe a formalism for studying spherically symmetric collapse of the massless scalar field in any spacetime dimension, and for any value of the cosmological constant Λ. The formalism is used for numerical simulations of gravitational collapse in four spacetime dimensions with negative Λ. We observe critical behaviour at the onset of black hole formation, and find that the critical exponent is independent of Λ.

It is now well established that gravitational collapse in spherical symmetry exhibits a phase transition-like critical behaviour, accompanied by self-similar behaviour of the matter field [1, 2, 3]. The basic formula determined numerically for the black hole radius near the threshold of black hole formation is [1].

$$R_{BH} \sim (a - a_*)^\gamma, \tag{1}$$

where a is an initial data parameter, and $a_* < a$ is its critical value. The critical value $a = a_*$ gives a naked singularity, and illustrates a violation of the cosmic censorship conjecture. The exponent γ is found to be "universal" for a fixed matter type in the sense that it is independent of the functional form and parameter values of the initial matter profile. However it varies with the type of matter field. For perfect fluids for example, γ depends on the parameter(s) in the equation of state[4].

A wide class of matter types yield two similar general features near criticality: the black hole radius scaling law (without a mass gap), and discrete or continuous self-similarity of the matter fields. (These are called Type II transitions. There are also transitions with a mass gap, refered to as Type I, where the field is either periodic or static, rather than self-similar.)

One of the questions that has not been studied to date is that of the dependence of the critical exponent on the cosmological constant Λ. We study this question in four spacetime dimensions. (The case of three dimensions is rather special in that there are no static black hole solutions unless the cosmological constant is negative; the constant may be scaled to unity without loss of generality so there is no need to study such dependence in this case [5, 6]). Our main result is that γ is independent of Λ.

A useful by-product of our analysis is confirmation of the utility of a new formalism for studying the collapse problem, in which the $d-$dimensional, spherically symmetric Einstein-scalar equations are rewritten as an effective 2−dimensional dilaton gravity theory[7]. All spacetime and Λ information in this approach is stored in the resulting dilaton potential. This feature permits writing a "universal" numerical code in which spacetime dimension and Λ appear as input parameters. In fact the formalism is general enough to permit simulations of mass and potential terms for the scalar field, with minimal changes to the code.

The reduced equations are written in double null coordinates, and a numerical method first used by Goldwirth and Piran[9], and refined by Garfinkle[10] is implemented on the resulting equations. The code is in principle capable of handling any spacetime dimension and cosmological constant value, although there are practical constraints. We describe below the main features of the formalism and numerical method. Further details appear in Ref. [8].

Einstein gravity with cosmological constant and minimally coupled scalar field in d spacetime dimensions is given by the action

$$S^{(d)} = \frac{1}{16\pi G^{(d)}} \int d^d x \sqrt{-g^{(d)}} \left[R(g^{(d)}) - \Lambda \right] - \int d^d x \sqrt{-g^{(d)}} |\partial \chi|^2. \tag{2}$$

Spherical symmetry is imposed by writing the metric $g_{\mu\nu}$ as

$$ds^2_{(d)} = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta + r^2(x^\alpha) d\Omega_{(d-2)}, \tag{3}$$

where $d\Omega_{(d-2)}$ is the metric on S^{d-2} and $\alpha, \beta = 1, 2$.

A useful form for the reduced action with this form of the metric is obtained by defining $l = (G^{(d)})^{n/2}$ and

$$\phi := \frac{n}{8(n-1)} \left(\frac{r}{l} \right)^n, \tag{4}$$

$$g_{\alpha\beta} := \phi^{2(n-1)/n} \bar{g}_{\alpha\beta}, \tag{5}$$

where $n \equiv d - 2$. Note that the ϕ is proportional to the area of an n-sphere at fixed radius r. With these definitions the reduced action becomes

$$S = \frac{1}{2G} \int d^2 x \sqrt{-g} \left[\phi R(g) + V^{(n)}(\phi, \Lambda) \right]$$

$$-\int d^2x\sqrt{-g}\, H^{(n)}(\phi)\,|\partial\chi|^2 \qquad (6)$$

where $G \equiv 2\pi n/(n-1)$, the scalar field has been rescaled $(\chi \to \chi/\sqrt{G^{(d)}})$ in order to make it dimensionless, and the overall factor of the $n-$sphere volume has been dropped from the action. In addition we have defined

$$H^{(n)}(\phi) \equiv \frac{8(n-1)}{n}\phi, \qquad (7)$$

$$V^{(n)}(\phi,\Lambda) \equiv \frac{1}{n}\left[\frac{8(n-1)}{n}\right]^{\frac{1}{n}}\phi^{\frac{1}{n}}$$
$$\times\left[\frac{n^2}{8}\left(\frac{8(n-1)}{n}\right)^{\frac{n-2}{n}}\phi^{-\frac{2}{n}} - l^2\Lambda\right] (8)$$

Now with the metric parametrized as

$$ds^2 = -2lg(u,v)\phi'(u,v)dudv \qquad (9)$$

the field equations are

$$\dot\phi' = -\frac{l}{2}V^{(n)}(\phi)g\phi' \qquad (10)$$

$$\frac{g'\phi'}{gH^{(n)}(\phi)} = 2G(\chi')^2 \qquad (11)$$

$$(H^{(n)}(\phi)\chi')^{\cdot} + (H^{(n)}(\phi)\dot\chi)' = 0, \qquad (12)$$

where prime and dot denote the v and u derivatives respectively.

The evolution equations may be put in a form more useful for numerical solution by defining the variable

$$h = \chi + \frac{2\phi\chi'}{\phi'}, \qquad (13)$$

which replaces the scalar field χ by h. The evolution equations become

$$\dot\phi = -\tilde g/2 \qquad (14)$$

$$\dot h = \frac{1}{2\phi}(h-\chi)\left(g\phi V^{(n)} - \frac{3}{2}\tilde g\right), \qquad (15)$$

where

$$g = \exp\left[4\pi\int_u^v dv\frac{\phi'}{\phi}(h-\chi)^2\right], \qquad (16)$$

$$\tilde g = l\int_u^v (g\phi'V^{(n)})dv, \qquad (17)$$

$$\chi = \frac{1}{2\sqrt\phi}\int_u^v dv\left[\frac{h\phi'}{\sqrt\phi}\right]. \qquad (18)$$

This is the final form of the equations used for numerical evolution. The spacetime parameters n and Λ appear only in the dilaton potential $V^{(n)}$ (8).

The numerical scheme uses a v discretization to obtain a set of coupled ODEs:

$$h(u,v) \to h_i(u), \qquad \phi(u,v) \to \phi_i(u). \qquad (19)$$

where $i = 0,\cdots,N$ specifies the v grid. Initial data for these two functions is prescribed on a constant v slice, from which $g(u,v), \tilde g(u,v)$ are constructed. Evolution in u ('time') is performed using the 4th. order Runge-Kutta method. The general scheme is similar to that used in [9], with some refinements used in [10]. This procedure was also used for the $3-$dimensional collapse calculations in [6].

The initial scalar field configuration $\chi(\phi, u=0)$ is most conveniently specified as a function of ϕ rather than r. (Recall that $\phi \propto r^n$.) This together with the initial arrangement of the radial points $\phi(v, u=0)$ fixes all other functions. We used the initial specification $\phi(0,v) = v$.

Most of the computations are for the initial scalar field of the Gaussian form

$$\chi_G(u=0,\phi) = a\phi\exp\left[-\left(\frac{\phi-\phi_0}{\sigma}\right)^2\right], \qquad (20)$$

with attention restricted to variations of the amplitude a. However we also used the "cosh" initial data

$$\chi_G(u=0,\phi) = a\cosh[b(\phi-\phi_0)]^{-2}, \qquad (21)$$

to study convergence of our code and to test universality with non-zero Λ.

The initial values of the other functions are determined in terms of the above by computing the integrals for g_n and $\tilde g_n$ using Simpson's rule.

The boundary conditions at fixed u are

$$\phi_k = 0, \quad \tilde g_k = 0, \quad g_k = 1. \qquad (22)$$

where k is the index corresponding to the position of the origin $\phi = 0$. (In the algorithm used, all grid points $0 \le i \le k-1$ correspond to ingoing rays that have reached the origin and are dropped from the grid). These conditions are equivalent to $r(u,u) = 0$, $g|_{r=0} = g(u,u) = 1$, and guarantee regularity of the metric at $r = 0$. For our initial data, ϕ_k and hence h_k are initially zero, and therefore remain zero at the origin because of Eqn. (15).

At each u step, the function

$$ah \equiv g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = -\frac{\dot\phi}{lg}, \qquad (23)$$

is observed. Its vanishing signals the formation of an apparent horizon. For each run of the code with fixed amplitude a, this function is scanned from larger to smaller radial values after each Runge-Kutta iteration, and evolution is terminated if the value of this function reaches 10^{-4}. (The code cannot continue much beyond this value.) The corresponding radial coordinate value is recorded as R_{ah}. In the subcritical case, it is expected that all the radial grid points reach zero without detection of an apparent horizon. This is the signal of pulse reflection. The results (a, R_{ah}) are collated by seeking a relationship of the form

$$R_{ah} \propto (a - a_*)^\gamma. \qquad (24)$$

The code was tested for grid sizes ranging from 2000 to 20000 points, and with the u and v step sizes ranging from 10^{-2} to 10^{-6}, for the two types of initial data used, as well as the vacuum case $\chi = 0$. These tests established that the code converges.

For most of the runs we used $\phi_0 = 1$ and $\sigma = 0.3$ for the Gaussian initial data, and $b = 5.0$, $\phi_0 = 0.5$ for the cosh data. We varied ϕ_0 in the Gaussian data from 0.1 to 20 (which also varies pulse width in the r-coordinate), to see if there was γ dependence. This is important to study because Λ sets a scale in the problem.

Our results for the two types of data (for $n = 2$) appear in Tables 1 and 2, which list the computed γ values for the corresponding (negative) values of Λ. The trend is clear: γ *does not depend on* Λ. The first value provides a crucial test of our formalism and code since it reproduces the known $\Lambda = 0$ value of the exponent [1].

Λ	γ
-0.001	0.370-0.375
-5	0.37-0.38
-10	0.37-0.39
-20	0.36-0.38

Table 1. Computed ranges of γ for Gaussian data ($n = 2$).

Λ	γ
-5	0.37-0.38
-20	0.36-0.38
-50	0.37-0.40

Table 2. Computed ranges of γ for cosh data ($n = 2$).

The ranges of γ in Tables 1 and 2 are arrived at by assessing our uncertainty in the determination of the critical amplitudes $a*$: the $a*$ values are determined to lie within certain domains, and the end points of these domains are used to determine the range of γ values. For example, for $\Lambda = -20$ and cosh data, we find a* lies in the range $6.00344 \times 10^{-3} - 6.00360 \times 10^{-3}$, which gives γ in the range indicated in Table 2. Our determination of the $a*$ window gets coarser with increasing Λ, which leads to the larger error bars on the γ values, as indicated in the tables.

Figure 1 is the ln-ln graph of the apparent horizon radius vs initial data amplitude, for $\Lambda = -5$. This is representative of the type of results from which the critical exponent ranges in Tables 1 and 2 are deduced.

Figure 2 shows plots of the apparent horizon function (23) and the scalar field h. This is typical of the type of data used to extract horizon radius information at the supercritical onset of black hole formation. These graphs show impending black hole formation at $\phi \sim 6.3 \times 10^{-6}$ ($r \sim 5 \times 10^{-3}$) for a supercritical amplitude with $\Lambda = -5$.
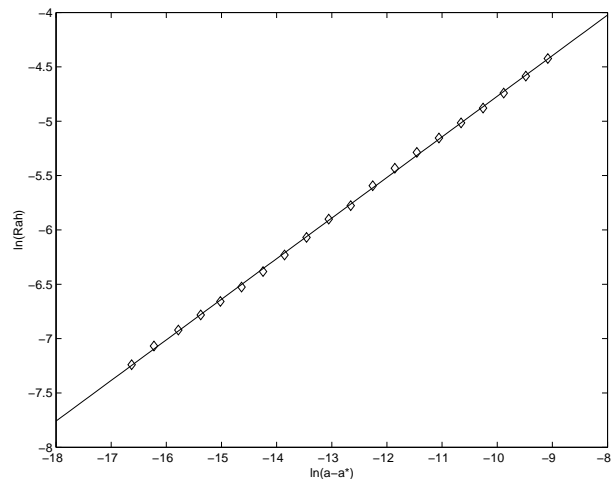

FIG. 1: Logarithmic plot of apparent horizon radius R_{ah} versus initial scalar field amplitude $(a - a_*)$ for $\Lambda = -5$. The line is the least squares fit to the points giving $\gamma = 0.3738$
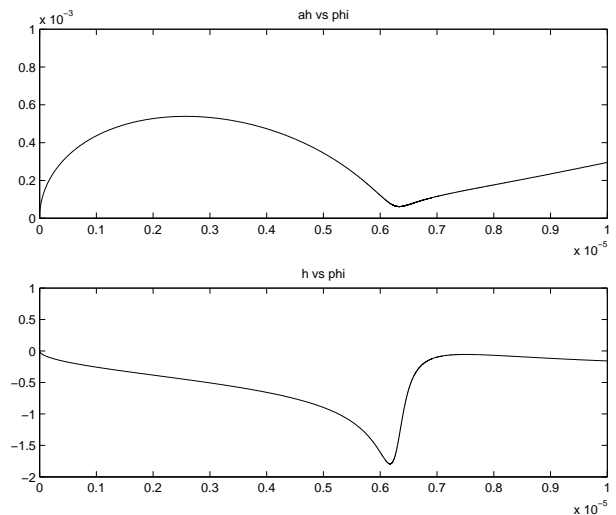

FIG. 2: Typical graphs of the apparent horizon function (23) and the scalar field h near black hole formation.

The dip in the apparent horizon function (Fig. 2) oscillates toward and away from the origin as it approaches the ϕ−axis, and appears to be the cause of the small oscillations about the least squares fit line in Figure 1. Although these latter oscillations have been noted in earlier work, it appears that they are the manifestation of the dip oscillations in the apparent horizon function (23) near criticality. This in turn is connected with the discrete self-similarity of the scalar field, which not surprisingly, is also manifested in the apparent horizon function. Figure 1 contains two complete oscillations, and may be used to obtain a rough estimate of the echoing period: it gives $\Delta \sim 3.5$. This value is close to the $\Lambda = 0$ value

$\Delta = 3.44$ computed in earlier work[1, 2, 3]. Thus, both the numbers γ and Δ associated with critical behaviour appear to be independent of Λ.

It is known, and verified again here, that γ is independent of initial data profiles and parameter values. We also find that for fixed Λ, critical exponents are independent of characteristic pulse width and starting location in relation to the Λ scale.

It is worth noting that in our equations Λ appears only in the potential term (8), where for sufficiently small ϕ, the Λ term is insignificant. This provides some analytical support of our results, since it suggests that the entire near critical part of the evolution, with typical ϕ range $0 - 10^{-5}$ (Fig. 2), is Λ independent. In four dimensions, $n = d - 2 = 2$, the first term in brackets in (8) is of order 10^5, which dominates the Λ term for all cases we consider. An extrapolation of this observation, beyond numerical reach, suggests that γ is Λ independent because sufficiently close to criticality, the ϕ term dominates the Λ term for any Λ (in an $\epsilon - \delta$ sense).

It is interesting to contrast this with the minimally coupled scalar field of mass μ, which also has a scale. Here two types of critical behaviour are observed at the threshold of black hole formation, depending on initial pulse width σ in comparison with μ^{-1} [12]: for $\sigma >> \mu^{-1}$ there is a mass gap, whereas for $\sigma << \mu^{-1}$ there is no mass gap and the exponent $\gamma \sim 0.378$ is computed[12].

There are some differences here worth emphasizing. The first is specific to our results, and the second is general: (i) We find no evidence of a mass gap for a range of values of pulse width and initial position in relation to the scale $1/\sqrt{-\Lambda}$. With no mass gap, the critical exponent is measured in the limit of small horizon values. In our simulations, the typical horizon size range at the onset of black hole formation is $10^{-2} - 10^{-4}$. By contrast, the smallest Λ scale the code can handle is $1/\sqrt{50} \sim 0.14$. Under these circumstances, it is not surprising that Λ has

no effect. (ii) Unlike μ, Λ appears in the stress-energy tensor independent of the scalar field. Therefore it is not unreasonable that Λ and μ give qualitatively different results.

For postive Λ we are not able to obtain accurate results for γ because of two competing effects in our procedure: since the grid is evolving, we observe an outward deSitter expansion of the grid, which confines the interesting features of the ingoing collapse to an ever shrinking region near the origin. Perhaps this case can better studied by the method of replacing lost grid points used in [10], and using the subcritical method of computing γ using the curvature scalar, introduced in [13]. Nevertheless, on the intuitive grounds mentioned, we expect that the sign of Λ will not change our results.

It is known that critical exponents may be computed via linear perturbation analysis of the critical solution [4, 11]. This has so far been done only for $\Lambda = 0$. For $\Lambda \neq 0$, it is reasonable to expect that both the critical solution and its perturbation equation depend on Λ, which after all is a parameter in the equations of motion. It would be useful to see how Λ drops out of this type of calculation, yielding a Λ independent exponent.

In summary, our numerical simulations of massless scalar field collapse in spherical symmetry in four dimensions, show that the critical exponent associated with the collapse is independent of (negative) Λ values. This result extends the scope of universality to include the cosmological constant, and suggests that including Λ with other matter types, such as the perfect fluid or the Yang-Mills field, will also not change the critical exponent.

[1] M. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[2] C. Gundlach, Living Rev. Rel. 2, 4 (1999).

[3] L. Lehner, Class. Quant. Grav. 18, R25-R86 (2001).

[4] D. Maison, Phys.Lett. **B366**, 82-84, (1996).

[5] F. Pretorius and M. Choptuik, Phys. Rev. **D62** 124012 (2000).

[6] V. Husain and M. Olivier, Class. Quant. Grav **18** L1-L10 (2001).

[7] J. Gegenberg, D. Louis-Martinez and G. Kunstatter, Phys. Rev. **D51** 1781 (1995); D. Louis-Martinez and G. Kunstatter, Phys. Rev. **D52** 3494 (1995).

[8] M. Birukou, V. Husain, G. Kunstater, E. Vaz and M. Olivier, Phys. Rev.**D65** 104036 (2002).

[9] D. Goldwirth and T. Piran, Phys. Rev. **D36**, 3575 (1987).

[10] D. Garfinkle, Phys. Rev. **D51**, 5558 (1995).

[11] T. Koike, T. Hara, S. Adachi, Phys. Rev. Lett. **74**, 5170(1995).

[12] P. R. Brady, C. M. Chambers and S. Goncalves, Phys. Rev. **D56**, 6057-6061 (1997).

[13] D. Garfinkle and G. Comer Duncan, Phys. Rev. **D58** (1998) 064024.